On fusion systems of blocks of finite groups and stable equivalences of Morita type

Dedicated to the 80th Birthday of Prof. Broué

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Basic conventions

- Assume throughout this talk that:
  - Any grp is finite;
  - p: fixed prime.
- $(K, \mathcal{O}, k)$: a complete p**-modular system**, where
  - K: a field with a complete discrete valuation v;
  - $\mathcal{O}$: the valuation ring of v;
  - k: residue field of $\mathcal{O}$ of char. p.

  Assume $(K, \mathcal{O}, k)$ is large enough for all grps and blocks involved and $\mathcal{O} = k$ is allowed.
- Blocks of $\mathcal{O}G$ and kG correspond bijectively, identified with primitive central idempotents of $\mathcal{O}G$.

G-algebras, Trace maps and Brauer maps

- G-alg. over $\mathcal{O}$ = $\mathcal{O}$-alg. with action of G as $\mathcal{O}$-alg. auto, denoted usually as conjugation ${}^g a$.
- interior G-alg. A = G-alg. given by grp hom. $G \to A^\times$.

A: G-alg, $P \leq G$.

- A^P: $\mathcal{O}$-subalg. of fixed points of P;
- **Trace map** $\mathrm{Tr}_P^G\colon\ A^P \to A^G,\ a \mapsto \sum_{t \in [G/P]} {}^t a$.
  $A_P^G := \operatorname{Im} \mathrm{Tr}_P^G$.
- **Brauer quotient**: $A(P) = A^P / \left(\sum_{Q<P} A_Q^P + J(\mathcal{O}) A^P \right)$.
- **Brauer map**: the natural surjection $\mathrm{Br}_P^A\colon\ A^P \to A(P)$.
- When $A = \mathcal{O}G$, $(\mathcal{O}G)(P) \cong kC_G(P)$ and
  $\mathrm{Br}_P^A\colon\ (\mathcal{O}G)^P \to kC_G(P)$ is just the truncation.

Brauer pairs[1]

Let b be a block of $\mathcal{O}G$. Alperin and Broué introduced a system of local data for b, *i.e.* (G,b)**-Brauer pairs** (Q, e_Q), where

- Q is a p-subgrp of G;
- e_Q is a block of $kC_G(Q)$ s.t. $e_Q \operatorname{Br}_Q(b) = e_Q$.

(G,b)-Brauer pairs satisfy Sylow-like properties:

- The set of all (G,b)-Brauer pairs form a G-poset;
- The maximal (G,b)-Brauer pairs form a G-conj. class whose first components are def. grps of b;
- Any (G,b)-Brauer pair is contained in a maximal (G,b)-Brauer pairs.

[1] J. L. Alperin and M. Broué, Local methods in block theory, Ann. Math. 110 (1979), 143–157.

Fusion systems

Fusion systems (F.S.), introduced by L. Puig[2], can be used to organize local data.
A F.S. $\mathcal{F}$ on a p-grp P is a category[3,4], whose

- objects are subgrps of P;
- hom set $\mathrm{Hom}_{\mathcal{F}}(Q,R)$ for any subgrps Q,R of P consists of injective grp homs satisfying
  - $\mathrm{Hom}_P(Q,R) \subseteq \mathrm{Hom}_{\mathcal{F}}(Q,R)$;
  - each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q,R)$ is the composition of an $\mathcal{F}$-iso. and an inclusion.
- A F.S. $\mathcal{F}$ on P is called *saturated* if every subgrp of P is $\mathcal{F}$-conjugate to a fully automized and receptive subgrp. (In F.S. $\mathcal{F}$, $\mathcal{F}$-isomorphism is usually called $\mathcal{F}$-conjugate.)

[2] L. Puig, Structure locale dans les groupes finis, Bull. Soc. Math. France Suppl. Mém., 47, 1976.

[3] M. Aschbacher, R. Kessar, and B. Oliver, Fusion systems in algebra and topology, London Math. Soc. Lecture Notes Series 391, Cambridge University Press, 2011.

[4] D. A. Craven, The theory of fusion Systems, Cambridge Studies in Advanced Mathematics 131, Cambridge University Press, 2011.

Alperin's fusion theorem

- $\mathcal{F}$: a F.S. on P.
- A subgrp $Q \leq P$ is called **fully $\mathcal{F}$-centralized** if $|C_P(Q)|$ is maximal among all $\mathcal{F}$-conjugates of Q

Theorem (Alperin's fusion theorem (weak form))

$\mathcal{F}$: a saturated F.S. on P. Then $\mathcal{F}$ is generated by $\mathrm{Aut}_{\mathcal{F}}(Q)$ with Q running over all fully $\mathcal{F}$-centralized subgrps of P.

F.S. afforded by grps

Let P be a p-grp.

- Assume G is a grp containing P as a Sylow p-subgrp. The F.S. $\mathcal{F}_P(G)$ is defined as below
    - objects are subgrps of P;
    - hom set for any subgrps Q, R of P is

$$\mathrm{Hom}_{\mathcal{F}_P(G)}(Q,R) = \{c_g \colon Q \to R \mid g \in G,\ {}^gQ \leq R\}.$$

- $\mathcal{F}_P(G)$ as above is saturated.
- Different choices of Sylow p-subgrps give iso. F.S.
- The smallest F.S. on P is $\mathcal{F}_P(P)$.

F.S. of blocks

b: a block of $\mathcal{O}G$ with a maximal (G,b)-Brauer pair (P, e_P).

- For any subgrp Q of P, there is a unique block e_Q of $kC_G(Q)$ s.t. $(Q, e_Q) \leq (P, e_P)$.
- The category $\mathcal{F}_{(P,e_P)}(G,b)$ of b w.r.t. (P, e_P) is defined as
  - objects are subgrps of P;
  - hom set for any subgrps Q, R of P is

$$\begin{aligned}&\mathrm{Hom}_{\mathcal{F}_{(P,e_P)}(G,b)}(Q,R)\\ =&\left\{c_g : Q \to R \mid g \in G,\, {}^{g}(Q,e_Q) \leq (R,e_R)\right\}.\end{aligned}$$

  Here, ${}^{g}(Q, e_Q) \leq (R, e_R)$ is equivalent to ${}^{g}e_Q = e_{{}^{g}Q}$.
- $\mathcal{F}_{(P,e_P)}(G,b)$ as above is saturated.
- Different choices of maximal Brauer pairs give iso. F.S.

Source idemp. and source alg (Puig)

b: a block of $\mathcal{O}G$ with a def. grp P.

- Any primitive idemp. i in $(\mathcal{O}Gb)^P$ s.t. $\mathrm{Br}_P(i) \neq 0$ is called a **source idemp.** of b w.r.t. P.
- If i is a source idemp. of b, then $i\mathcal{O}Gi$ is called a **source alg.** of b, which is Morita equiv. (M.E.) to $\mathcal{O}Gb$.

i: a source idemp. of b.

- There is a unique maximal (G, b)-Brauer pair (P, e_P) s.t. $\mathrm{Br}_Q(i)e_P \neq 0$.
- For any subgrp Q of P, there is a unique block e_Q of $kC_G(Q)$ s.t. $\mathrm{Br}_Q(i)e_Q \neq 0$, and thus $(Q, e_Q) \leq (P, e_P)$.
- So a source idemp. w.r.t. P fixes a maximal (G, b)-Brauer pair (P, e_P) and a F.S. of b on P.

Source algs and fusion systems

Puig showed that F.S of blocks are determined by source algs[5]. We use the statement given by Linckelmann[6].

Theorem (Puig)

b: a block of $\mathcal{O}G$ with a def. grp P and a source idemp. i in $(\mathcal{O}Gb)^P$. Denote by $\mathcal{F}$ the F.S. of b determined by i. Let $Q, R \leq P$.

- *Every ind. direct summand of $i\mathcal{O}Gi$ as an $(\mathcal{O}Q, \mathcal{O}R)$-bimod is iso. to $\mathcal{O}Q \otimes_{\mathcal{O}S} {}_{\varphi}\mathcal{O}R$, where $S \leq Q$, $\varphi\colon S \to R$ in $\mathcal{F}$.*
- *If $\varphi\colon Q \to R$ is an $\mathcal{F}$-iso. with R fully $\mathcal{F}$-centralised, then ${}_{\varphi}\mathcal{O}R$ is iso. to a direct summand of $i\mathcal{O}Gi$ as an $(\mathcal{O}Q, \mathcal{O}R)$-bimod.*
- *Q: fully $\mathcal{F}$-centralised, $\varphi \in \mathrm{Aut}(Q)$. Then $\varphi \in \mathrm{Aut}_{\mathcal{F}}(Q)$ iff ${}_{\varphi}\mathcal{O}Q$ is iso. to a direct summand of $i\mathcal{O}Gi$ as an $(\mathcal{O}Q, \mathcal{O}R)$-bimod.*

Thus by Alperin's fusion theorem, $\mathcal{F}$ is determined by the $(\mathcal{O}P, \mathcal{O}P)$-bimod. structure of $i\mathcal{O}Gi$.

[5] L. Puig, Local fusions in block source algebras, J. Algebra 104 (1986), no. 2, 358–369.

[6] Theorem 8.7.1 in: M. Linckelmann, The Block Theory of Finite Group Algebras. Vol. II. Cambridge University Press, 2018.

Nilp. blocks

Theorem (Frobenius theorem)

Assume P is a Sylow p-subgrp of G, then G is p-nilpotent (i.e. there is a normal subgrp H of G of p'-order such that $G = HP$) iff $\mathcal{F}_P(G) = \mathcal{F}_P(P)$.

Nilpotent blocks are Motivated by the above Frobenius theorem.

Definition (Broué-Puig)

[a] Assume b is a block of $\mathcal{O}G$ with a maximal (G, b)-Brauer pair (P, e_P). If $\mathcal{F}_{(P,e_P)}(G, b) = \mathcal{F}_P(P)$, then b is called **nilpotent**.

[a] M. Broué, L. Puig, A Frobenius theorem for blocks, Invent. Math. 56:2 (1980), 117–128.

Nilp. blocks (continued)

Puig determined the structure of source algs of nil. blocks. Here, an $\mathcal{O}P$-mod. V is permutation (**endopermutation**) if V ($\mathrm{End}_{\mathcal{O}}(V) \cong V \otimes_{\mathcal{O}} V^*$) has a P-stable basis.

Theorem (Puig)

[a] *Assume b is a nilp. block of $\mathcal{O}G$ with a def. grp P. Then $\exists$ an endoperm. $\mathcal{O}P$-mod. V s.t.*

$$i\mathcal{O}Gi \cong \mathrm{End}_{\mathcal{O}}(V) \otimes_{\mathcal{O}} \mathcal{O}P \quad \text{as } P\text{-algs.}$$

In particular, $\mathcal{O}Gb$ is M.E. to $\mathcal{O}P$.

[a] L. Puig, Nilpotent blocks and their source algebras, Invent. Math. 93 (1988), 77–116.

Stable equivalent of Morita type

Stable equivalent of Morita type (S.E.M.T), introduced by Broué, is a generalization of M.E.

Definition

[a] Let A, B be $\mathcal{O}$-algs. If $\exists$ (A, B)-bimod M and (B, A)-bimod N s.t.

$$M \otimes_B N \cong A \oplus X, \quad N \otimes_A M \cong B \oplus Y$$

for some proj. $A \otimes_R A^{\mathrm{op}}$-mod. X and some proj. $B \otimes_R B^{\mathrm{op}}$-mod. Y, we said that A, B are S.E.M.T. via M and N.

[a] M. Broué, Equivalences of blocks of group algebras, in: Finite Dimensional Algebras and Related Topics, Kluwer, 1994, pp.1–26.

Remark

- S.E.M.T. induces equivalence between stable categories.
- When A, B are symmetric and indecomposable (*e.g.*, blocks of finite grps), we can choose $N \cong M^*$ and ind.

S.E.M.T and nil. blocks

By Puig's result above, nil. blocks are M.E. to grp algs of defect grps. Puig asked: whether the converse holds? Later, Puig gave a positive answer under weaker condition

Theorem (Puig)

[a] *Assume $\mathcal{O}$* ***is of char.*** 0. *If the block $\mathcal{O}Gb$ is S.E.M.T to a nilp. block, then b is nilp: S.E.M.T between blocks preserves nilpotency.*

[a] L. Puig, On the local structure of Morita and Rickard equivalences between Brauer blocks, Progress in Mathematics, 178, Birkhäuser Verlag, Basel, 1999.

- Puig's proof relies heavily on his theory of non-injective induction of interior G-algebras.
- Recently, a modular-theoretical proof of Puig's theorem above is given[7] using some arguments of Linckelmann.
- "$\mathcal{O}$ **is of char.** 0" because we need a result of Weiss[8].

[7] C. Li, On stable equivalences of Morita type and nilpotent blocks, J. Algebra 665 (2025), 243–252.

[8] A. Weiss, Rigidity of p-adic p-torsion, Ann. Math. 127 (1988), 317–332.

S.E.M.T v.s. F.S. and defect grps

Theorem (Puig)

[a] *S.E.M.T with endoperm. sources preserve fusion systems and in particular defect grps.*

[a] L. Puig, On the local structure of Morita and Rickard equivalences between Brauer blocks, Progress in Mathematics, 178, Birkhäuser Verlag, Basel, 1999.

- Under general S.E.M.T (not necessarily with endoperm. sources), the preservation of defect grps fails at least over k, since $\exists$ counterexamples even for the Modular Isomorphism Problem (MIP)[9]: if P_1 and P_2 are two p-grps s.t. $kP_1 \cong kP_2$ as k-algebras, then whether we have $P_1 \cong P_2$?
- Isomorphism Problem for p-grps over $\mathcal{O}$ of char. 0 indeed holds[10,11].

[9] D. García-Lucas, L. Margolis, Á. del Río, Non-isomorphic 2-groups with isomorphic modular group algebras, J. Reine Angew. Math. 783 (2022), 269–274.

[10] K.W. Roggenkamp, L.L. Scott, Isomorphisms of p-adic group rings, Ann. Math. (2) 126 (1987), 593–647.

[11] A. Weiss, Rigidity of p-adic p-torsion, Ann. Math. 127 (1988), 317–332.

S.E.M.T and F.S.

Consider general S.E.M.T (not necessarily with endoperm. sources). Let $\mathcal{F}$ be a F.S. on a p-grp P.

- $R \leq P$ is called **strongly $\mathcal{F}$-closed** if $\varphi(S) \leq R$ for any $S \leq R$ and any $\varphi\colon S \to P$ in $\mathcal{F}$; in particular, $R \trianglelefteq P$.
- When R is a strongly $\mathcal{F}$-closed subgrp of P, a **quotient F.S.** $\mathcal{F}/R$ can be defined.

Theorem (L., arXiv: 2509.24795, to appear in Sci. China. Math.)

b_1 (b_2): block of $\mathcal{O}G_1$ ($\mathcal{O}G_2$) with defect grp P_1 (P_2); M: an $(\mathcal{O}G_1b_1, \mathcal{O}G_2b_2)$-bimodule inducing a stable equivalence. Assue $R \leq P_1 \times P_2$ is a vertex of M and set $R_1 = R \cap P_1, R_2 = R \cap P_2$. There is source idemp. $i_1 \in (\mathcal{O}G_1b_1)^{P_1}$ ($i_2 \in (\mathcal{O}G_2b_2)^{P_2}$) determining F.S. $\mathcal{F}_1$ ($\mathcal{F}_2$) on P_1 (P_2).

- *$|P_1| = |P_2|$ and both projections from R to P_1 and P_2 are surjective, thus $|R_1| = |R_2|$ and $P_1/R_1 \cong P_2/R_2$.*
- *R_1 is strongly $\mathcal{F}_1$-closed in P_1 iff R_2 is strongly $\mathcal{F}_2$-closed in P_2.*
- *$\mathcal{F}_1/R_1 \cong \mathcal{F}_2/R_2$ when the equivalent conditions above hold.*

Remark

- $|P_1| = |P_2|$ follows from that S.E.M.T preserves stable Grothendieck grps[12] [13].
- The assertion that both projections from R to P_1 and P_2 are surjective is a result of Puig[14], while an alternative proof is given[15].
- The main tool to compare F.S. is the above Puig's result that F.S. is determined by source alg.
- Since $R_1 = R_2 = 1$ is equivalent to having endoperm. sources, we recover Puig's results on F.S. in this case.
- We do not know whether indeed R_1 is strongly $\mathcal{F}_1$-closed in P_1 and R_2 is strongly $\mathcal{F}_2$-closed in P_2.

[12] M. Broué, Equivalences of blocks of group algebras, in: Finite dimensional algebras and related topics, Kluwer (1994), 1–26.

[13] C. Xi, Stable equivalences of adjoint type, Forum Mathematicum 20 (2008), 81–97.

[14] L. Puig, On the local structure of Morita and Rickard equivalences between Brauer blocks, Progress in Mathematics, 178, Birkhäuser Verlag, Basel, 1999.

[15] C. Li, On stable equivalences of Morita type and nilpotent blocks, J. Algebra 665 (2025), 243–252.

Thank You!